

Humanizing Rational Numbers in Middle School

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Abstract

This paper aims to re-examine the curricular pedagogy and framework of math education, particularly concerning rational numbers in middle school. The introduction presents an overview of sequential pathways and statistical data in U.S. mathematics education, followed by a summary of rational number concepts through the lens of constructivist learning theory and the author's positionality surrounding these topics. The literature review explores attributes of successful organizations and systems, followed by an analysis of mathematics education in the U.S. as an organizational system, and the evolution of the language of mathematics to its current state. Implications for learning are then considered from the following perspectives: students, teachers, and curricula. Considering national and anecdotal data, math learning theory, the rational number domain as a sequential step to algebra, and the physio-cultural evolution of our number system, arguments and strategies for humanizing rational numbers for students are presented. The author hypothesizes that a dual-pronged approach of (a) increasing students' sense of agency and belonging in the math classroom by offering students a way to recognize themselves as successive participants in the global journey of math; and (b) providing base-ten contextualized algorithmic lessons that demystify common operational stumbling blocks within the rational number domain, holds the potential to appease both sides of the systemic math education debate (accessibility vs. rigor) and lay the groundwork for students' future academic opportunities by increasing their math achievement.

Keywords: math education, rational numbers, middle school, place value, teacher education

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Introduction

Several studies identify success in algebra as the gateway to higher Education, progression in STEM (science, technology, engineering, and math) fields, and even economic opportunity (Darling, 2010; Miller & Kimmel, 2012). It is not an exaggeration to say that algebra marks a watershed moment in the life of a student. It is impossible to advance in mathematics without it. However, according to the Mathematical Association of America, each year, roughly 50 percent of students fail to pass college algebra with a grade of “C” or better (Liston & Getz, 2019). Similar to most systemic dysfunctions, by the time problems arise, the issues have been present for some time. Troubling statistics plague the nation's K-8 schools. For example, standardized math assessment data from 2019 indicate a null change from 2017 and “proficient” scores for only 41% of fourth graders and 34% of eighth graders (National Center for Education Statistics, 2019). This downward trending effect was compounded in subsequent data by pandemic-era learning (National Assessment of Educational Progress (NAEP), 2022). Costly ed-tech interventions and ambitious reform proposals have yielded little fruit (Doroudi et al., 2019; Knee, 2016).

Algebra represents a shift from quantifying the known world to quantifying a realm of possible worlds. It repurposes every operational component in the vast web of the base ten universe for a whole new machine, one that is adaptable and predictive rather than discrete. Algebraic math is base ten math version 2.0. The domain that precedes algebraic numbers is rational numbers. Rational numbers build on students’ prior learning of whole numbers to allow them to think in terms of ratio or one whole number compared to another whole number. The term “ratio” refers to the quantitative relationship between two values, showing how many times one value contains or is contained within the other. (Simpson, 2002). Rational numbers are defined by their ability to be written in *ratio*, i.e., fraction form. The rational number domain encompasses most number concepts and operations that do not involve variables. These concepts can include but are not limited to fractions, decimals, percents, rates, ratios, unit conversions, exponents (squares and cubes), square roots and cube roots, scientific notation, and all four number operations within these concepts.

The idea that changing one or the other number within a ratio changes its overall quantitative value is a logical and necessary stepping stone to algebraic functions, where the input value or independent variable determines the output value or dependent variable. Put another way, rational numbers represent a student’s first function, where the number of times one value contains the other determines its overall output value.

Rational numbers are usually introduced in third grade with fractions and studied almost exclusively by the time students reach sixth grade. While many reforms have focused their attention on algebra, owing to its academic gateway status, the reality is that students without a solid understanding of rational numbers will not be able to successfully transition to algebraic number sense, regardless of the robustness of the algebra-based intervention. Therefore, middle school math education represents the cornerstone of a K-12 student's math journey because rational number sense is solidified in middle school.

While several learning theories are referenced throughout this paper, middle school students' knowledge of and fluency with rational numbers is primarily explored through a constructivist framework. Constructivism is the belief that groups construct knowledge with and for one another, including a shared culture with shared artifacts imbued with meaning (Ertmer & Newby, 2013). Our number system represents the single most shared cultural artifact the globe has ever produced, yet most math curricula ignore this potentially humanizing feature. Constructivism further contends that new information is "constructed" in the mind, requiring learners to retrieve prior information in the process.

Civil rights leader and activist Angela Davis said, "Radical simply means 'grasping things at the root'" (Davis, 1984, p. 14). Whether Davis intended it or not, this quotation has mathematical and social connotations. For example, the radical symbol in the problem $\sqrt{64}$ asks the student, *what is the one number (the root) needed to grow a square of 64?* A wealth of information must precede this problem to truly grasp its meaning, symbolically and conceptually, and to know that a dimension of eight (by another dimension of eight) grows a square of 64.

The implication of Davis' statement can and should also be applied to math. The tyranny of the present often dictates that teachers focus on upcoming tests and standardized assessments through algorithmic practice siloed in grade bands. However, math is sequential. New skills are constructed on the foundation of prior concepts and skills. If students are to truly take ownership of mathematics, they must be encouraged to grasp it at the root, not just once but as a continual practice.

Many of the theories in this paper originate from questions that date back to 2013. That was the year my daughter started middle school, and her math scores plummeted. Usually a great student, neither of us could identify precisely how it started. Slowly, she stopped raising her hand and turned her focus toward subjects that seemed more hospitable. I recognized the forces that were quietly but firmly ushering my daughter away from math because they had done the same to me when I was her age. Despite a love for numbers, patterns, and art as an elementary student,

by the time I reached my college counselor's office, I was choosing the least rigorous courses that would satisfy my B.A. degree math credit requirements. It was disheartening to realize that the intervening years had produced few if any, additional means of accessibility.

Given this historical context, I was not interested in reviewing algorithms that would help my daughter get through the next set of homework problems. I recognized that she had lost her footing in math in a more holistic sense, and my goal was to help her regain it in the long run. Not knowing where to start, I sought the beginning. The beginning of math. That was the day it dawned on me: I was subconsciously laboring under a lifelong assumption that math predated humans. Math seemed intrinsically divine, abstractly mystical. Throughout my academic life, I experienced math as a code that was there for the taking if only I were smart enough to "get it."

Of course, numbers, the values and movements inherent to the cosmos, predate humans, but the language of math does not. It was forged by ancient cultures all over the world, born out of a basic human drive to explore and communicate (Howe, 2018; O'Connor & Robertson, 2000; Smith & Ginsburg, 1937; Yong & Se, 2004). Over a decade and hundreds of middle school math students later, Davis' epithet still hangs on my classroom wall.

From my perspective as a middle and graduate school classroom teacher, I intend to show how the multiple systems that govern rational number study in the U.S., from the conflicting stances of Higher Education to pedagogical shortcuts, are counterproductive to teacher preparation and learning acquisition, respectively. I propose that the best way forward is to go back to the source, to how the language of math was constructed in the first place, and to the foundation of that language, the base ten place value chart.

Literature Review

This literature review aims to give an overview of the research regarding successful organizations and systems, math education in the U.S. as an organizational system, and the language of math as it was organized by humankind.

Successful Systems and Organizations

There is a wealth of evidence to support the idea that the most successful systems and organizations (regardless of their size or area of focus) are the ones that can straddle and equally value paradoxical elements such as individuality and corporation, innovation and specialization, and myth and fact (Argyris, 1957; Barnes et al., 2013; Bolman & Deal, 2013; Pugh, 1971).

One observable hallmark of ineffectual organizations is that no effort is made to merge the individual and organizational processes involved in self-actualization (Barnes et al., 2013).

Individual self-actualization and corporate self-actualization, and the erroneous sacrifice of the former for the sake of the latter, are characterized as being consistently devoted to maximizing human potential for the sake of the organization (Argyris, 1957). When self-actualization is sacrificed, personal motivation wanes, and productivity follows.

A similar effect is produced when specialization is favored over innovation. While people may initially appreciate the gratification involved in specialized roles, eventually, as the work becomes more rote or routine, motivation is lost. Well-intentioned but ultimately anti-innovative advice like “fit the man to the job and the job to the man” created situations where workers lost a sense of connection to their jobs in terms of interest and satisfaction (Pugh, 1971).

While few would doubt the importance of dealing with facts within organizations, a landmark study conducted over 50 years ago discovered the importance of shared ritual, mythology, and symbolic meaning within organizations through the role of religion. Religion, in this case, Catholicism, created more systemic commonalities in companies across differently industrialized countries than between companies in similarly industrialized countries without a shared religion (Pugh, 1971). Shared myths are foundational elements of human culture, fostering a sense of belonging and identity within communities. They provide a framework for understanding the world and shaping collective values, beliefs, and behaviors. Myths and symbols are essential for societies to transmit knowledge, preserve traditions, and reinforce social cohesion across generations (Bolman & Deal, 2013). This research suggests that within any human-driven system, symbolic and mythic elements are as crucial as literal and factual ones.

Compounding the issue of instilling paradoxical elements into systems and organizations is the human tendency to view problems as frame-bound rather than reality-bound. Bolman and Deal (2013) assert that many of us passively acquiesce to solutions to problems as they are presented, thus seldom encountering opportunities to discern the degree to which our preferences are constrained by framing rather than grounded in reality. True organizational success requires the continual infusion and balance of opposing forces, yet we are often presented with evidence that compels us to delegitimize one or the other force. Frame-bound solutions that seek to eliminate a crucial, opposing force will only perpetuate the cycle of dysfunction.

Math Education in the U.S.

The U.S. math instruction as an organizational system has been mired in a frame-bound, opposing force-eliminating cycle. For the sake of simplicity, I will call one force rigor-centered

(factual, specialized, technical) and the other accessibility-centered (mythologized, innovative, artistic).

From a systemic perspective, historically, math curricula and instruction in the U.S. have suffered from a lack of accessibility-centered force. District initiatives focused on standardized assessments subsume individuality, algorithmic specialization overrides innovation, and facts displace myth. When considering the quantitative nature, indeed the minutiae and the breadth, of K-12 math instruction, it is easy to understand how educators, and society in general, found themselves in this curricular predicament.

Interventions that aim to increase accessibility tend to focus less on content and more on culturally responsive teaching practices and methods for sharing authority in the math classroom (Holland, 2022; Kang et al., 2018; Langer-Osuna et al., 2020; Selbach-Allen et al., 2020). Proponents of these strategies insist they can move the needle regarding academic achievement. However, those results remain to be seen. Many schools, districts, and even states have instituted alternative math pathways such that students can “opt out” of traditional math and enroll in courses that are more akin to math appreciation to satisfy their math credit graduation requirements (California Department of Education, n.d.; Liston & Getz, 2019). While a case could be made for allowing alternative pathways in specific, individualized contexts, the problem with this strategy as a wholesale solution is that it does nothing to address the concerns of the rigor-minded set and generally contradicts what countless teachers profess daily as truth in terms of growth mindset and the universal ability to learn math to high levels (Boaler et al., 2021; Selbach-Allen et al., 2020). Policies that adjust content rather than delivery give the subtextual impression that our math achievement scores are so low as to be insurmountable.

Dissenters of these policies, many of them higher education STEM professors, claim that they detract from and diffuse the academic rigor needed to practice and learn math at high levels. Math education as a system is continually oscillating between these opposing forces of maintaining rigor and increasing accessibility. The opposing sides’ mistrust of and focus on the other’s methods have created unprecedented learning gaps and have cost American students greatly.

These two warring schools of thought will continue to cost the U.S. in terms of student gains as long as they are frame-bound instead of reality-bound. To effectively restructure math education, we must be willing to break out of traditional educational frameworks and the detrimental belief that there is only one correct pedagogical focus (rigor or accessibility). If any academic field should know about opposing and balancing forces, the ingredients for every

formula and equation, it is math. It is time we take the data to heart and reframe math instruction from a reality-bound perspective.

When considering the cut-and-dried nature of math, indeed the fact that there is only one correct answer to every problem, one can understand how, in our effort to enhance the math potential of our students, we lost the forest for the trees. Unlike English Language Arts and other academic subjects within the humanities, which offer obvious ways for students to personally connect to texts and historical events in the process of self-actualization, math self-actualization must be more intentional. We have become so focused on the specialized subject of math (the job) and the students who learn it (the man) that we have inadvertently forgone any chance at real curricular innovation.

In his seminal work juxtaposing two struggling Chicago schools, “Organizing Schools for Improvement,” Bryk et al. (2010) noted that while reading gains could still be made in schools despite weak supports, no school lacking in support showed improvement in mathematics. The language of numbers has yet to be contextualized for students in the same way the language of words has. It is hidden behind a shroud of inaccessible cryptology, a chasm that appears to increase with each subsequent school year.

Reframing math education from a reality-bound perspective requires that we put down our chalk for a minute and go back to the beginning, the beginning of math. It is easy to see the apparent ways math is quantitative; it is the language of numbers. However, most ignore the qualitative elements intrinsic to math, the way that human beings across time and cultures shaped it into a useful, shared artifact (O’Connor & Robertson, 2000; Smith & Ginsburg, 1937; Yong & Se, 2004).

Few things in life are more bluntly factual than balancing an equation to solve for x . While we may not be able to incorporate mythology into every equation, we can help students grasp the shared symbolism that gave rise to our number system. The language of math quite literally depends on symbols.

The Language of Mathematics

Much like tree leaves and ocean shells contain math in the form of fractals and numerical sequences, the language of math contains human fingerprints, of those who brought it into existence and those who continue to use it. From this perspective, every subsequent equation or operation is also tied inextricably to human connectedness and ingenuity.

Imagine you are an alien from outer space and stumble upon the symbol “3.” Would you have any idea what it meant? There is nothing inherent to “3” that makes us know that it represents a quantity of three. This exercise allows us to imagine how students construct mathematical understanding through a process called concrete-pictorial-abstract or CPA.

In the concrete stage, a student can hold up three fingers or collect three blocks to communicate a value in a one-to-one ratio. In the pictorial stage, the student can draw a quantity, for example, three flowers, to show understanding and begin to adjust to the idea of representing value through symbolism. The final stage is abstract, here, the student uses the symbol “3” to mean three of something. This seemingly mundane transition is not just an arrival at a procedural destination but a rite of passage inducting the learner into a vast, intricate, globally shared web of collective, symbolic meaning.

We have been teaching students dozens of math symbols, from numbers and operations to exponents, variables, and formulas without any of the ritualistic shared meanings that brought them into being in the first place.

Much of that ritual can be found in a weathered and decaying booklet, “Numbers and Numerals: A Story Book for Young and Old,” published in 1937 by the Columbia Teachers’ College (Smith & Ginsburg, 1937). Full of imagery and diagrams, from the eloquent stick system of China to the pragmatic finger numerals of Europe, it is clear that humans and numbers have a long and storied relationship. It is also clear that our current base ten positional system is arguably the single greatest global achievement of all time (Howe, 2018; O’Connor & Robertson, 2000; Smith & Ginsburg, 1937; Yong & Se, 2004). Most cultures began with hash marks, or a unary system, which represented value in a 1:1 ratio. As civilizations became more sophisticated, symbols denoting larger values in varying ratios, such as the Romans’ L for 50 or 1:50, emerged. Eventually, humans kept the idea of symbols like the Romans, but unlike the Romans, they devised individual symbols for 1-9 based on ancient Sanskrit (Smith & Ginsburg, 1937). Thus, Hindu-Arabic numerals were employed. These were given further meaning by formalizing our biological affinity for 10 through the symbols’ placement on an invisible 1:10 ratio board of ones, tens, and hundreds, aka place value. One hundred years later, around 600 CE, the concept of 0 as a numeral caught on, which exponentially increased the new system’s functionality and paved the way for decimals (O’Connor & Robertson, 2000; Smith & Ginsburg, 1937).

Implications: Constructing Mathematical Literacy in the Rational Number Domain

This section aims to provide an overview within a constructivist framework for how mathematical literacy is thwarted or achieved. Student, teacher, and curricular perspectives are considered.

Student Learning

When considering rational number comprehension from the student's perspective, it is essential to recognize that rational numbers represent the second domain of number sense they acquire. The first number sense students develop is natural numbers, sometimes called counting numbers, e.g., 1, 2, 3, 4, and so on. In this domain, students learn to order and compare numbers, as well as compute values using the four primary operations of addition, subtraction, multiplication, and division. Over time, they develop an intuitive sense of how these numbers behave, e.g., adding and multiplying numbers gives a greater answer, and subtracting and dividing numbers gives a lesser answer (De Keersmaecker et al., 2022; Van Hoof et al., 2020). When students begin to study rational numbers, they often erroneously apply numerical reasoning intrinsic to their natural number sense, called natural number bias (Van Hoof et al., 2020).

Due to natural number bias, numbers may appear larger to students that are not, e.g., $15.2 > 5.1346$ (more digits do not mean a greater number) and $\frac{1}{8} < \frac{1}{2}$ (larger digits do not mean a greater number). Operations follow suit in this inverse world, e.g., $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ (the product is smaller than its factors) and $0.6 \div 0.2 = 3$ (the quotient is larger than the dividend and divisor). It is not enough for students to take the teacher's word for the correct answer and how to get there. They need to know why this inverse world exists and how to navigate it themselves.

Furthermore, many sixth-grade students begin using the set of integers, including negative numbers. From anecdotal data, the most common error of writing a negative number line is writing it from greatest to least, e.g., -1, -2, -3, -4, and so on. This response contains natural number bias in the form of a directional error. Unaddressed, these errors persist and are exacerbated with negative rational numbers, e.g., on a number line with -6 on the left and -5 on the right, many students mark the midpoint as -6.5 when the correct answer is -5.5.

Consider a student entering the rational number domain who is taught to "move the decimal" rather than the digits to solve various problems. This means that they must (a) know that as we read numbers left to right, their place values go from greatest to least (the opposite of how they learned to count) (Howe & Epp, 2008); (b) understand place value's multiplicative structure, that a digit *moves left when multiplied* by ten and *right when divided* by ten; and (c)

fluidly and simultaneously translate this knowledge to instead move the decimal *right to multiply* by ten and *left to divide* by ten. This is not something we should ask our math learners to do. It is time to consider the true cost of teaching this deceptively convenient and shortcut practice. It is akin to teaching a child to play chess and then telling them to move the board instead of the pieces.

As a middle school math teacher, I have let students ponder the same questions I had: Why does math exist? How did it become a language? What other number systems were used, and what is it about base ten that made it stick? Alongside the organizational ritualism that involves in tying meaning to symbols lies the learning theory known as Constructivism. As previously mentioned, Constructivism is the belief that groups construct knowledge with and for one another, including a shared culture with shared artifacts imbued with meaning (Ertmer & Newby, 2013).

Additionally, Constructivism contends that new information is “constructed” in the mind, requiring learners to retrieve prior information in the process. The Vygotskian theory of Social Constructivism embraces the idea that we first learn new information in a social context from a more knowledgeable other before internalizing that information for repeated use over time. Phrases such as “move the decimal” essentially skip the entire social construction process and move immediately to internalization. Once students have fully developed rational number sense and have moved on to algebraic number sense, these phrases may become part of their internalized dialogue. However, relying on them during the learning process can delay learning or create a faulty foundation for subsequent knowledge construction.

Exploring the role of our biological affinity for ten in the global formation of our number system leverages the learning theory known as embodied cognition, which contends that the physical elements of the human body significantly shape the workings of the human mind (American Psychological Association, n.d.). This theory is further extrapolated to mathematics in what Nathan and Walkington (2017) called the theory of grounded and embodied mathematical cognition or GEMC. GEMC suggests employing physical actions and gestures to comprehend concepts associated with science, technology, engineering, and mathematics. At the same time, embodied cognition as it pertains to STEM fields has been researched, as well as the historical origins of our number system; both fields stop short of making the explicit curricular link between students’ deep understanding of place value and the physio-cultural origins of our number system. This gap in academic research needs to be acknowledged and addressed.

The goal of humanizing rational numbers seeks to build on the existing literature, which deals with common pitfalls students encounter while constructing rational number understanding, such as natural number bias and rote algorithm errors. The cognitive process known as transference is critical for students transitioning from whole to rational number sense (Ertmer & Newby, 2013). When the Common Core State Standards uses one of its eight Standards for Mathematical Practice to encourage students to “look for and make use of Structure,” they are leveraging the idea of transference (Common Core State Standards Initiative, n.d.). Transference occurs when situations characterized by identical or similar features facilitate the transfer of knowledgeable behaviors across shared elements (Ertmer & Newby, 2013). Like transference, the cognitive process known as adaptive expertise is a highly valued feature of mathematical thinking.

In contrast to routine expertise, adaptive expertise is characterized by interconnected procedural and conceptual knowledge that can be flexibly employed in unfamiliar situations (McMullen et al., 2021). Adaptive expertise is noted for increasing procedural flexibility with whole numbers, rational numbers, and linear algebra. Students who were taught rote, siloed algorithms (such as *keep it, change it, flip it* for fraction division), and procedural shortcuts (such as *moving the decimal* for multiplication and division) are denied the opportunity to socially construct the expertise necessary for future mathematical domains. Intuitive place value knowledge is a prerequisite for adaptive expertise and transference to occur.

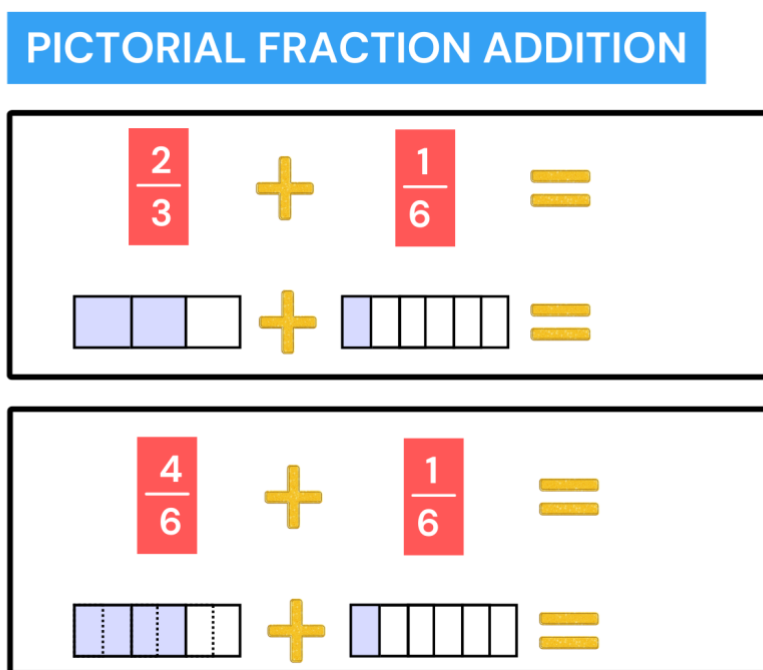
Similarly, Agarwal and Bain (2019) advocated for multiple learning strategies in “Powerful Teaching: Unleash the Science of Learning.” Among these strategies are two practices called retrieval and interleaving. Retrieval involves the skilled extraction of information from students’ knowledge reservoirs, in contrast to the act of cramming information into them. Interleaving involves mixing closely related topics and encouraging students to draw comparisons and distinctions between the two (Agarwal & Bain, 2019).

Recall the CPA (concrete-pictorial-abstract) method of constructing mathematical understanding to understand the importance of retrieval. Rather than stand up at the chalkboard and drill algorithms into students’ heads, a CPA-trained teacher encourages students to use prior stages of learning to personally and actively construct conceptualization and consequently imbue meaning to the third and final abstract algorithmic stage. A typical example (shown in Figure 1) highlights the conceptual underpinnings of equivalent fractions when adding or subtracting fractions with unlike denominators. When a student draws a bar model to represent $\frac{2}{3} + \frac{1}{6}$ she is revisiting the pictorial stage to imbue meaning to the abstract stage. In this example, the

importance of only adding quantities of the same relative (ratio) size is illustrated in the pictorial representation of the equivalent fractions $\frac{2}{3}$ and $\frac{4}{6}$. As a student internalizes this concept, she will be able to reason abstractly to find an equivalent fraction, e.g., $\frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$. Retrieval in action looks like encouraging elementary students to explore concrete (with math manipulatives like base ten blocks) and pictorial stages for every new abstract skill, from adding fractions to subtracting across zeros to the multiplication algorithm to long division and so on.

Figure 1

CPA Example of Abstract Fraction Addition Tethered to Pictorial Representation



The learning theory of interleaving, or mixing up closely related topics, is leveraged as students compare and contrast whole number sense and operations with rational number sense and operations (Agarwal & Bain, 2019). Interleaving in the context of whole and rational numbers might look like asking students why the product of rational numbers is often smaller than its factors when the reverse is true for whole numbers or what might have caused the directionality error in a fictitious student's negative number line. Encouraging students to revisit natural numbers and comprehend their own natural number bias enables them to assess and contrast prior foundational knowledge regularly. This allows them to incorporate a whole new number sense, rational, into their cognitive repertoire. To quote a famous childhood organization's motto, *make new friends, but keep the old; one is silver, and the other's gold*.

Furthermore, the strategy of “friendly numbers” allows students to leverage their intuitive natural number sense to understand rational number word problems. For example, when given a word problem with rational numbers, many students (and adults) experience confusion and anxiety upon seeing non-natural numbers, such that it prevents them from making a plan to solve the problem. Consider the following word problem: *Anna has 8 liters of juice which she would like to pour evenly into $\frac{1}{5}$ -liter cups. How many cups can Anna fill?* The inclusion of the rational number $\frac{1}{5}$ prevents students from using their natural number intuition to solve this problem.

However, if we temporarily substitute a “friendly” or whole number for $\frac{1}{5}$, such as 2, we can easily decide which operation to use. *Anna has 8 liters of juice, which she would like to pour evenly into 2-liter cups. How many cups can Anna fill?* This problem now becomes one that an average third-grader could solve mentally. Based on anecdotal data, when using this strategy with sixth-graders, they are quite excited to shout "Four!" When asked what operation they used, they think back to pouring 8 liters into 2-liter cups and say, "Division!" The interleaving of whole and rational number sense has provided a procedural roadmap. This inter-domain scaffold enables them to replace the original number in the problem and solve it correctly: $8 \div \frac{1}{5} = 40$ cups.

Teacher Preparation

Humanizing rational numbers for first-year middle schoolers must also address the issue of teacher preparedness in terms of rational number teaching. Vanhoof et al. (2018) assert that by the end of elementary school, only a limited subset of learners fully understands the intricate structure of rational numbers. Much of this difficulty arises from the fact that teachers’ understanding of rational numbers is often very limited (Howe, 2020; Ma & Ma, 1999; Thanheiser, 2009; Van Hoof et al., 2018). While insightful and robust, this body of research frequently diagnoses the issue and offers potential solutions. However, further research is required to see if the prescribed solutions are effective. Many K-8 math teachers were taught math in a similar, rote way—devoid of symbolic and concrete meaning. We must attend to them as their students’ primary source of math information.

Many teachers lack what researcher Liping Ma called PUFM, or a profound understanding of fundamental mathematics (Howe, 2020; Ma & Ma, 1999). In her seminal report, researcher Eva Thanheiser (2009) found that only 20% of the preservice teachers in her study could reliably conceive of the reference units for each digit in a three-digit number. Thanheiser concluded that in order to facilitate student understanding of numbers and algorithms, teachers need more than the ability to perform the algorithms. Given that Social Constructivism requires the presence of a

“more knowledgeable other,” it is essential that Thanheiser’s conclusion be taken into account when it comes to professional development regarding place value and rational numbers.

Curricular Implications

Akin to the cornerstone that is middle school math education, the decimal point is the keystone of our base ten positional number system. It lives in a fixed location between the one’s and the tenth’s place. Nevertheless, popular curricula and online platforms such as [khanacademy.com](https://www.khanacademy.com) teach decimal movement, some as young as fifth grade (Houghton Mifflin Harcourt, 2020). Given the stages of constructing mathematical understanding on the part of the students and the need for PUFM on the part of the teachers, it is hard to justify this method of math instruction. Considering the cultural and linguistic diversity in today’s classrooms, it becomes even more challenging to justify this methodology and its obvious toll on students’ ability to build a solid rational number sense to support future study. Countless students enter algebra classrooms thinking they can “move the decimal” or “add a zero” when multiplying by ten (Howe & Epp, 2008). Similarly, students learning rational numbers are often told to “keep it, change it, flip it” to divide fractions. While momentarily expedient, ultimately, uncontextualized algorithmic practices sever the mathematical meanings from the symbols and structures created to represent them.

Children who are taught math in this rote way can be successful for a time. Many students can memorize dozens of siloed tricks (untethered to base ten understanding) which temporarily provide correct answers (Irwin, 2001; Lortie-Forgues & Siegler, 2017; McMullen et al., 2021). However, this methodology is failing them when they try to learn algebra. Consider that we read (text and numbers) from left to right, and we begin teaching math by counting from least to greatest. Forty percent of the Kindergarten Common Core State Standards and substandards are in the domain of “counting and cardinality” (Common Core State Standards Initiative, n.d.). However, from left to right, digits in our place value system go from greatest to least (hundreds, tens, ones). This makes sense from a utility standpoint; as adults, we want to know about the 100 dollars before the tens and ones, but it is prudent for educators to think about this multi-directionality in terms of student learning, especially those with directional learning difficulties. Furthermore, most students today have very little interaction with physical money. Grounding concepts such as regrouping across place values through the ritualistic acts of making change and finding sums or differences with whole and fractional monetary units, once the strongholds in a teacher’s contextualizing toolbox, are verging on obsolete (Irwin, 2001; McMullen et al., 2021).

There are myriad ways to leverage our natural inclination to base ten in pursuit of plasticity and fluency with rational numbers, from sums of ten in early elementary to scientific notation in high school, which leads to deeper conceptualization. This is especially true for percent, decimal, and fraction (i.e., rational number) domains where many of our most academically vulnerable middle schoolers, e.g., first-years, live. Notably, sixth grade is also where the strand “Numbers and Operations in Base Ten” stops appearing in the Common Core (Common Core State Standards Initiative, n.d.). The shift from using whole number sense to building rational number sense is a critical stepping stone to building algebraic number sense. Rational number knowledge is a strong predictor of more advanced mathematical learning, yet many students’ math struggles begin with the introduction of rational numbers (Thompson & Smith, 2017). Causes of this are commonly attributed to students’ natural number bias and teachers’ lack of PUFM.

Conclusion

Given our country’s continually declining math achievement scores and the importance of building a solid rational number sense for future study, there is evidence to support an examination and restructuring of the organization of math education as it pertains to curricular instruction, particularly where the rational number domain is concerned.

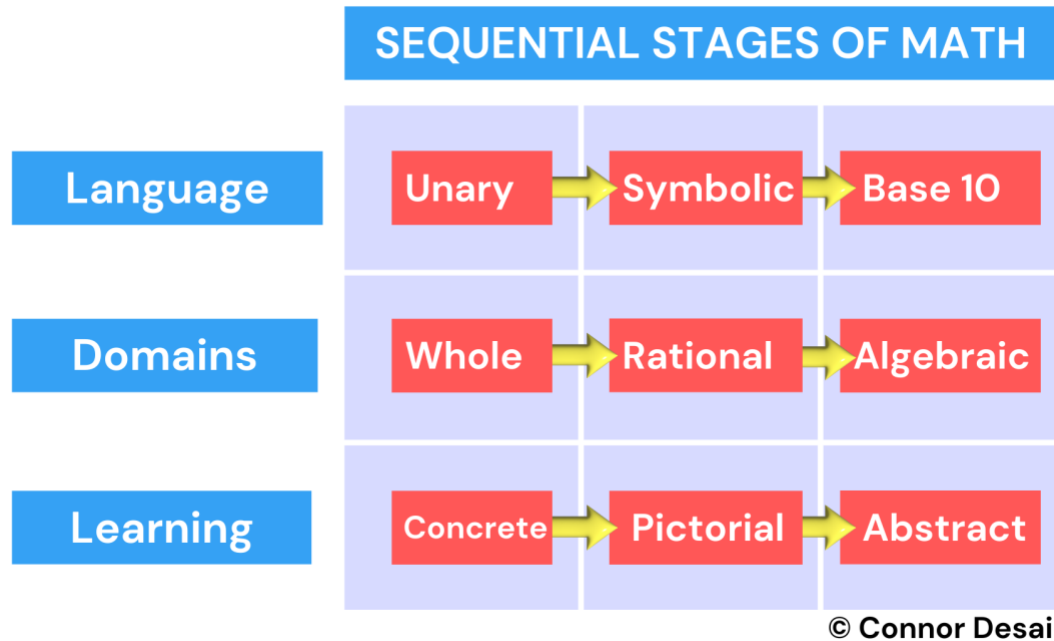
As Hosun Kang et al.’s (2018) work illuminates, middle school plays a critical role in shaping young individuals’ perceptions of themselves in relation to STEM subjects, influencing their long-term engagement with these fields. Ultimately, students need to feel successful in math (i.e., arrive at the correct answer and know *why* their computations worked) to feel confident in mathematics learning spaces (Thompson & Smith, 2017). The goal of decolonizing and humanizing rational numbers is more than a curricular enhancement. A lack of intuitive base ten place value knowledge is a form of illiteracy. Every new mathematical skill must be explicitly tethered to its shared meaning, its concrete, humanistic base ten roots. This is what it means to “get” math.

Each round of CPA stages, like each student’s journey toward constructing mathematical understanding, is a microcosm of humanity’s shared construction of our number system. There is an observable symmetry as students practice and master each number domain, from whole (natural) to rational to algebraic (see Figure 2). These symmetrical, sequential math learning processes, like our base ten positional system, are grounded in our biological evolution. Human ontogeny, the development of an individual organism within its lifetime, parallels phylogeny, the evolutionary history of the human species. Each progressive stage, starting from the yolk sac to gills and embryonic tail, mirrors the evolutionary progression of our ancestors, from single-cell

amoebas to fish and reptilian forms. All steps are necessary to arrive in human form, just as all steps are necessary to arrive at mathematical literacy.

Figure 2

Chart of Sequential Stages of Mathematical Language and Literacy Construction



Drawing back the curtain on the origins of math and contextualizing lessons to their base ten roots has the potential to address the needs of both sides of the math education debate. Rigor can be reinstated with the mathematical literacy that stems from base ten teaching. A working knowledge of humanity’s shared labor in creating the number system will improve accessibility in terms of students’ sense of agency and identity in mathematics learning spaces and perhaps dispel the pervasive but thinly disguised belief in the United States that white male students from affluent backgrounds inherently possess the “math gene” once and for all (Wibneh, 2016).

Eradicating the damage of harmful stereotypes, rote teaching methods, and poor mathematics achievement requires a restructured, systemic solution, one that invites children to identify themselves as successive participants in the journey of math.

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